# *Systèmes entrée-sortie non linéaires et applications en audio-acoustique*

## **Séries de Volterra**

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Ecole Thématique ”Théorie du Contrôle en Mécanique”
2019

# Vito Volterra [1860 (Ancona) - 1940 (Roma)]

**Mathématicien et physicien italien** connu pour ses travaux sur les **équations intégro-différentielles, la dynamique des populations**, ...
Prédécesseur de Fréchet et Banach, il est considéré comme l'un des fondateurs de l'analyse fonctionnelle.
Il s'oppose au régime fasciste (1922) et refuse les honneurs académiques par conviction politique. Il vit alors en grande partie à l'étranger et revient à Rome peu avant sa mort.

Royal Society (1910) - Royal Society of Edinburgh (1913)
Un cratère de lune porte son nom.

# Avant de commencer, quelques rélénces bibliographiques


- V. Volterra. *Theory of Functionnals and of Integral and Integro-Differential Equations*. (Dover Pub., 1959).
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- W. J. Rugh. *Nonlinear System Theory, The Volterra/Wiener approach*. (The Johns Hopkins University Press, Baltimore, 1981).
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- F. Lamnabhi-Lagarrigue. *Analyse des Systèmes Non Linéaires*. (Editions Hermès, 1994).
- S. Boyd and L. Chua. Fading memory and the problem of approximating nonlinear operators with Voltera series. (*IEEE-TCAS* , 32(11):1150–1161, 1985).
- M. Hasler. Phénomènes non linéaires. (*Ed. Ecole Polytechnique Fédérale de Lausanne*, 1999).
- F. Bullo. Series expansions for analytic systems linear in control. (*Automatica*, 38:1425–1432, 2002).
- Doyle *et al.* Identification and Control Using Volterra Models. (*Springer*, 2002).
- Hélie & collaborators. Quelques articles personnels joints (2003-2019).

# Plan

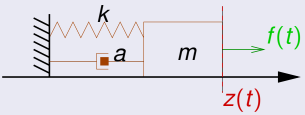
1. Préambule
2. Séries de Volterra : généralités
3. Calcul des noyaux de Volterra d'un système différentiel
4. Exercices et applications en audio-acoustique
5. Convergence
6. Extension en dimension infinie et application
7. Conclusion

# Plan

# Outline

# 1. AN EXAMPLE: Mass-Spring-Damper system

**Problem** (at rest for $t < 0$)




$$m z''(t) + a z'(t) + k z(t) = f(t)$$

**Find the trajectory $z(t)$ with respect to the force $f(t)$**

**System with input ($u$) / output ($y$)**

$u := f \longrightarrow$ External representation (closed-form ?) $\longrightarrow y := z$

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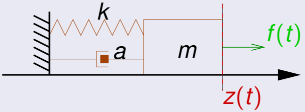


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## State-space representation: $\mathbf{x}(t) = [z(t), z'(t)]^T$, $\mathbf{x}(0) = [0, 0]^T$

$$\underbrace{\begin{bmatrix} z'(t) \\ z''(t) \end{bmatrix}}_{\mathbf{x}'(t)} = \underbrace{\begin{bmatrix} 0 & 1 \\ -k/m & -a/m \end{bmatrix}}_{\boldsymbol{A}} \underbrace{\begin{bmatrix} z(t) \\ z'(t) \end{bmatrix}}_{\mathbf{x}(t)} + \underbrace{\begin{bmatrix} 0 \\ 1/m \end{bmatrix}}_{\boldsymbol{B}} \mathbf{u}(t)$$

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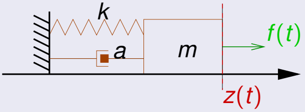


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$$\text{Eq.}: \boldsymbol{x}'(t) = \boldsymbol{A}\,\boldsymbol{x}(t) + \boldsymbol{B}\,\boldsymbol{u}(t)$$

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$$\boldsymbol{y}(t) = [1, 0]\ \boldsymbol{x}(t) = \boldsymbol{C}\ \boldsymbol{x}(t)$$

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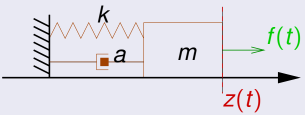


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$$\boldsymbol{y}(t) = \int_0^t h(\tau)\, u(t-\tau)\, \mathrm{d}\tau$$

**Convolution(/filtering) with the impulse reponse**

$h(\tau) = \boldsymbol{C}\, \mathrm{e}^{\boldsymbol{A}\tau} \boldsymbol{B}\, 1_{\tau \geq 0}$

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# 2. AN EXAMPLE: in the LAPLACE domain

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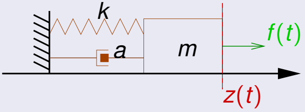


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$$\boldsymbol{Y}(s) = H(s)\, \boldsymbol{U}(s)$$

**Transfer function (/filter)**

$$H(s) = \boldsymbol{C}\,(s\,I_2 - \boldsymbol{A})^{-1} B$$

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$$\begin{aligned} \boldsymbol{Y}(s) &= [1, 0] \; \boldsymbol{X}(s) \\ &= \boldsymbol{C} \; \boldsymbol{X}(s) \end{aligned}$$

# Outline

# 3. What about nonlinear systems ?

## Input/Output nonlinear differential system (state x)

$$
\begin{aligned}
\mathbf{x}'(t) &= \mathbf{F}\big(\mathbf{x}(t), \mathbf{u}(t)\big) \\
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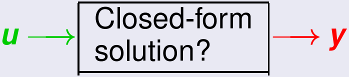
\end{aligned}
$$




In general, NO ! But...

## Linear case

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## I/O relation: linear filter

Kernel: $h(t) = \mathbf{C} e^{\mathbf{A}t} \mathbf{B} + \mathbf{D}$

Trsf. fct: $H(s) = \mathbf{C}(s\mathbf{I} + \mathbf{A})^{-1}\mathbf{B} + \mathbf{D}$

**Interests:** control, spectral analysis, identification, simulation, etc.

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$\boldsymbol{u} \longrightarrow$ Closed-form solution? $\longrightarrow \boldsymbol{y}$

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## Weakly nonlinear case

$\boldsymbol{F}$, $\boldsymbol{G}$: power series expansions *around equilibrium point 0 (nonzero coeff. at order 1)*

## Example: a nonlinear spring

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## I/O relation: Volterra series

**Interests:** idem!

# 4. From a qualitative point of view...

**A few comparisons:**

| Case | closed-form sol. w.r.t. input | distortions | self-oscillations bifucations, chaos |
|---|---|---|---|
| General | no | yes | yes |
| Volterra | yes | yes | no |
| Linear | yes | no | no |

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## Interest of Volterra series:

- **Natural distortions** for high amplitudes
- Possible extensions to **partial differential equations**
- Audio-acoustics: **large dynamics (/fortissimo)**

# 5. What is the idea ? (regular perturbation method)

**For a Weakly Nonlinear System ...**

$$\boldsymbol{x}'(t) = \boldsymbol{F}\big(\boldsymbol{x}(t), \boldsymbol{u}(t)\big) \qquad \boldsymbol{F}(X,U) = \sum_{m,n} \frac{D_{m,n}\boldsymbol{F}(0,0)}{m!n!}(X,\ldots,X,U,\ldots,U)$$

$$\boldsymbol{y}(t) = \boldsymbol{G}\big(\boldsymbol{x}(t), \boldsymbol{u}(t)\big) \qquad \boldsymbol{G}(X,U) = \sum_{m,n} \frac{D_{m,n}\boldsymbol{G}(0,0)}{m!n!}(X,\ldots,X,U,\ldots,U)$$

**Consider the input as a perturbation of the system.**
**Mark it with $\eta \in (0,1)$: $\boldsymbol{u}(t) = \eta\, v(t)$**

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(i) Introduce $\boldsymbol{x}(t) = \sum_n \eta^n \boldsymbol{x}_n(t)$ and $\boldsymbol{y}(t) = \sum_n \eta^n \boldsymbol{y}_n(t)$

(ii) Inject these series expansions in the system equations

(iii) Sort equations w.r.t. $\eta^n$

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(iv) Solution: Each $x_n$ is a **multiple convolution** of $n$ repeated versions of the input and a computable multivariate kernel

$\longrightarrow$ **Volterra kernel**

# Plan

# Outline

# Time domain

- Definition and examples
- A convergence criterion
- Non-uniqueness of kernels
- Remark on time-varying systems

# Volterra series: definition

## Definition

A system $\xrightarrow{u} \boxed{\{h_n\}} \xrightarrow{y}$ is defined by the Volterra series $\{h_n\}_{n\geq 1}$ if

$$y(t) = \underbrace{\sum_{n=1}^{+\infty}}_{\text{Sum}} \underbrace{\int_{\mathbb{R}^n} h_n(\tau_1, \dots, \tau_n)\, u(t-\tau_1) \dots u(t-\tau_n)\, \mathrm{d}\tau_1 \dots \mathrm{d}\tau_n}_{\text{of multiple convolutions}}$$

*(with several possible functional settings)*

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## Example

- Linear filters: $h_n = 0, \quad \text{if } n \geq 2.$

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$$= \sum_{n=1}^{+\infty} \alpha_n \left(u(t)\right)^n \qquad \text{Power series}$$

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## A system is **causal** if

$$\tau < 0 \quad \Rightarrow \quad h(\tau) = 0 \qquad \text{(linear)}$$

$$\exists k \text{ s.t. } \tau_k < 0 \quad \Rightarrow \quad h_n(\tau_1, \dots, \tau_k, \dots, \tau_n) = 0 \qquad \text{(Volterra)}$$

# A convergence criterion (1/2) (see e.g. [Boyd,1984])

## RECALL: definition of a Volterra series

$$x(t) = \sum_{n=1}^{+\infty} x_n(t) \textbf{ with } x_n(t) = \int_{\mathbb{R}^n} h_n(\tau_1, \ldots, \tau_n)\, u(t-\tau_1) \ldots u(t-\tau_n)\, \mathrm{d}\tau_1 \ldots \mathrm{d}\tau_n$$

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**Bounded Input Bounded Output (BIBO) result**
$(\|u\|_\infty = \sup_{t\in\mathbb{R}} |u(t)|)$

$$|x_n(t)| = \left| \int_{\mathbb{R}^n} h_n(\tau_1, \dots, \tau_n)\, u(t-\tau_1) \dots u(t-\tau_n)\, \mathrm{d}\tau_1 \dots \mathrm{d}\tau_n \right|$$

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$$\leq \int_{\mathbb{R}^n} \left|h_n(\tau_1, \dots, \tau_n)\right| \underbrace{\left|u(t-\tau_1)\right| \dots \left|u(t-\tau_n)\right|}_{\leq \|u\|_\infty \dots \|u\|_\infty} d\tau_1 \dots d\tau_n$$

$$\|x_n\|_\infty \leq \underbrace{\int_{\mathbb{R}^n} \left|h_n(\tau_1, \dots, \tau_n)\right| d\tau_1 \dots d\tau_n}_{=\|h_n\|_1 \quad (L^1\text{-norm})} \left(\|u\|_\infty\right)^n$$

# A convergence criterion (1/2) (see e.g. [Boyd,1984])

## RECALL: definition of a Volterra series

$$x(t) = \sum_{n=1}^{+\infty} x_n(t) \text{ with } x_n(t) = \int_{\mathbb{R}^n} h_n(\tau_1, \dots, \tau_n)\, u(t-\tau_1) \dots u(t-\tau_n)\, \mathrm{d}\tau_1 \dots \mathrm{d}\tau_n$$

## Bounded Input Bounded Output (BIBO) result

$\left(\|u\|_\infty = \sup_{t\in\mathbb{R}} \left|u(t)\right|\right)$

$$\begin{aligned}
|x_n(t)| &= \left| \int_{\mathbb{R}^n} h_n(\tau_1, \dots, \tau_n)\, u(t-\tau_1) \dots u(t-\tau_n)\, \mathrm{d}\tau_1 \dots \mathrm{d}\tau_n \right| \\
&\leq \int_{\mathbb{R}^n} \left|h_n(\tau_1, \dots, \tau_n)\right| \underbrace{\left|u(t-\tau_1)\right| \dots \left|u(t-\tau_n)\right|}_{\leq \|u\|_\infty \dots \|u\|_\infty} \mathrm{d}\tau_1 \dots \mathrm{d}\tau_n \\
\|x_n\|_\infty &\leq \underbrace{\int_{\mathbb{R}^n} \left|h_n(\tau_1, \dots, \tau_n)\right| \mathrm{d}\tau_1 \dots \mathrm{d}\tau_n}_{= \|h_n\|_1 \quad (L^1\text{-norm})} \left(\|u\|_\infty\right)^n
\end{aligned}$$

**Hence,** $\|x\|_\infty \leq \sum_{n=1}^{+\infty} \|x_n\|_\infty \leq \sum_{n=1}^{+\infty} \|h_n\|_1 \left(\|u\|_\infty\right)^n.$

# A convergence criterion (2/2) (see e.g. [Boyd,1984])

**Gain bound function $\varphi$**

Define $\varphi(z) = \sum_{n\geq 1} \|h_n\|_1 \, z^n$ with convergence radius $\rho$ at $z = 0$.

**Theorem (BIBO result)**

**If** $\|u\|_\infty < \rho$, **then** the Volterra series expansion of $x$ is normally convergent and

$$\|x\|_\infty \leq \varphi\big(\|u\|_\infty\big) < +\infty.$$

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Moreover, the **truncation error is bounded**:

$$\left\| \sum_{n=N+1}^{+\infty} x_n \right\|_\infty \leq \sum_{n=N+1}^{+\infty} \|h_n\|_1 \big(\|u\|_\infty\big)^n$$

# Non-uniqueness of Volterra kernels

**Remark:**

Permuting variables $\tau_k$ in $y(t) = \sum_{n=1}^{+\infty} \int_{\mathbb{R}^n} h_n(\tau_1, \ldots, \tau_n)\, u(t-\tau_1) \ldots u(t-\tau_n)\, \mathrm{d}\tau_1 \ldots \mathrm{d}\tau_n$ leaves the output $y$ unchanged.

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**Example**

$h_2(\tau_1, \tau_2)$, $h_2(\tau_2, \tau_1)$, but also $\alpha h_1(\tau_1, \tau_2) + (1-\alpha) h_2(\tau_2, \tau_1)$ define the same Input-Ouput system.

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**Symmetrical versions of Volterra kernels $SYM[h_n]$ are unique**

$$\mathrm{SYM}\left[h_n\right](\tau_1, \ldots, \tau_n) = \frac{1}{n!} \sum_\pi h_n(\tau_{\pi(1)}, \ldots, \tau_{\pi(n)})$$

Other unique versions (triangular kernels, regular kernels) are also available.

# Remark on time varying systems

**A definition is also available for time-varying systems:**

$$y(t) = \sum_{n=1}^{+\infty} \int_{\mathbb{R}^n} g_n\left(t, \theta_1, \dots, \theta_n\right) u(\theta_1) \dots u(\theta_n) \, \mathrm{d}\theta_1 \dots \mathrm{d}\theta_n$$

**Time-Invariant (TI) case and link with kernels $h_n$**

TI case: $y(t) = \sum_{n=1}^{+\infty} \int_{\mathbb{R}^n} h_n(\tau_1, \dots, \tau_n) u(t-\tau_1) \dots u(t-\tau_n) \, \mathrm{d}\tau_1 \dots \mathrm{d}\tau_n$

Kernels $g_n$ of a TI system are such that ($\theta_k = t - \tau_k$)

$$g_n\left(t, t-\tau_1, \dots, t-\tau_n\right) = h_n(\tau_1, \dots, \tau_n)$$

does not depend on $t$

**Causal kernels $g_n$**

$\exists k$ s.t. $\theta_k > t \Rightarrow g_n(t, \theta_1, \dots, \theta_k, \dots, \theta_n) = 0$

# Outline

# Laplace(/Fourier) domain and analogies with linear systems

**Laplace domain (or Fourier domain with $s = 2i\pi f$)**

Transfer function: $H(s) = \int_{\mathbb{R}} h(\tau) e^{-s\tau} \, d\tau$ (lin.)

Transfer kernel: $H_n(s_{1:n}) = \int_{\mathbb{R}^n} h_n(\tau_{1:n}) e^{-(s_1\tau_1 + \ldots + s_n\tau_n)} \, d\tau_1 \ldots d\tau_n$ (Volt.)

denoting $(s_{1:n}) = (s_1, \ldots, s_n)$ and $(\tau_{1:n}) = (\tau_1, \ldots, \tau_n)$.

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**Causal stable system: NO poles (and NO singularities)**

of $H(s)$ for $\Re\mathrm{e}(s) > 0$ (linear)

of $H_n(s_{1:n})$ for $\Re\mathrm{e}(s_k) > 0$ (Volterra)

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| | | |
|---|---|---|
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| of $H_n(s_{1:n})$ | for $\Re e(s_k) > 0$ | (Volterra) |

**Input/Output relation** $u \to$ system $\to y$

| | | |
|---|---|---|
| Transfer function: | $Y(s) = H(s)\, U(s)$ | (lin.) |
| Transfer kernel: | more complex | (Volt.) |

*(next part: use interconnection laws)*

# A result on periodic signals

## Analytic input signal $u(t) = ae^{i\omega t}$

$$u(t) = ae^{i\omega t} \longrightarrow \boxed{\{h_n\}} \longrightarrow y(t) = \sum_{n=1}^{+\infty} a^n H_n(i\omega, \ldots, i\omega)\, e^{in\omega t}$$

## Periodic input signals / Fourier series

$$u(t) = \sum_{k=-\infty}^{+\infty} u_k\, e^{ik\omega t} \longrightarrow \boxed{\{h_n\}} \longrightarrow y(t) = \sum_{k=-\infty}^{+\infty} y_k\, e^{ik\omega t}$$

with $y_k = \sum_{n=1}^{+\infty} \sum_{\substack{k_1,\ldots,k_n=-\infty \\ k_1+\ldots+k_n=k}}^{+\infty} u_{k_1} \ldots u_{k_n} H_n(ik_1\omega, \ldots, ik_n\omega)$

## Distortion coefficient for $u(t) = a\cos(\omega t)$

$D(a, \omega) = \sum_{n=2}^{+\infty} |y_n|^2 / |y_1|^2$ : closed-form solution w.r.t. $a$, $\omega$, $H_n$.

# In summary:

**A Volterra series ...**

- catches distortions (memory combined with nonlinearities)
- sorts the nonlinear responses w.r.t. the degree $n$ of homogenous contributions of $u$
- generalizes the convolution principle
- can be described by transfer kernels in the frequency domain (as filters).
- is usually non unique but uniquely defined versions are available (useful for identification purposes)

# Plan

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